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Harmonic 1-Forms on the Stable Foliation

François Ledrappier

Abstract. We study cohomology classes of Hölder continuous closed leafwise 1-forms on the stable foliation of an Anosov geodesic flow. Each class contains a harmonic 1-form and is determined by its periods. Asymptotic quantities are computed in terms of the Pressure function defined by the geodesic flow.

0. Introduction

Let (M, g) be a closed Riemannian manifold with negative sectional curvature, let $\mathcal{W}^s$ denote the stable foliation of the unit tangent bundle SM, and consider the differential complex of sections of real p-forms on $T\mathcal{W}$ which are smooth along the leaves of the foliation and such that all jets are globally Hölder continuous. The cohomology is trivial except in degree one where a closed 1-form α is exact if and only if

$$\int_\gamma \alpha = 0$$

for every closed curve γ which remains in the same leaf.

In this paper are studied some properties of cohomology classes of closed 1-forms. Our main result is that in each cohomology class there is a unique harmonic 1-form. We also define an "asymptotic cycle" and an "asymptotic energy" on cohomology classes, naturally associated with the leafwise Laplacean. In fact these properties are related to asymptotic properties of the leafwise heat kernel, which were established in [**L3**]. Using an idea in [**LJ**], we are able to express the above asymptotic

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quantities through the thermodynamical formalism of the geodesic flow.

I. Notations and results

1.1 Harmonic 1-forms

Let (M, g) be a closed Riemannian manifold with negative sectional curvature, SM the unit tangent bundle endowed with some smooth metric, $\mathcal{W}^s$ the stable foliation on SM. To define $\mathcal{W}^s$ recall that the geodesic flow ϕ_t is defined on SM by

$$\phi_t v = (\gamma_v(t), \dot{\gamma}_v(t))$$

where $t \to \gamma_v(t)$ is the geodesic in M defined by

$$(\gamma_v(0), \dot{\gamma}_v(0)) = v.$$

The stable leaf $W^s(v)$ is the set of w in SM such that there is a real $b(v, w)$ with

$$d(\phi_s v, \phi_{s+b(v,w)} w) \to 0 \text{ as } s \to +\infty.$$

The set $W^s(v)$ is a smoothly embedded manifold in SM and the function $w \to b(v, w)$ is smooth on $W^s(v)$. The stable manifolds $\{W^s(v), v \in SM\}$ form a Hölder continuous foliation of SM. Furthermore all jets of the function $w \to b(v, w)$ on $W^s(v)$ are also globally Hölder continuous on SM (see e.g. [**An**], [**HPS**]).

We consider in this paper stable forms, i.e. sections of the bundle of exterior forms on $T_v W^s(v)$. Denote d_s the leafwise differential operator on stable forms. The projection $w \to \gamma_w(0)$ defines a local diffeomorphism between $W^s(v)$ and M. Use this diffeomorphism to lift the metric g on M to a metric g_s on each stable manifold $W^s(v)$. Denote δ_s the codifferential operator on stable forms associated to g_s. In particular for a stable 1-form $\alpha, \delta_s \alpha$ is minus the g_s-divergence of the vector field $\alpha^\sharp$ in TW^s associated to α by the g_s-duality. Denote Δ_s the Laplacean $\Delta_s = -(\delta_s d_s + d_s \delta_s)$ and a stable form α is called harmonic if $\Delta_s \alpha = 0$. When acting on 0-forms, Δ_s is the leafwise Laplace-Beltrami operator defined on leafwise smooth functions. By [**G**], [**L2**], there is a unique Δ_s harmonic probability measure μ on SM, i.e. such that $\int \Delta_s u d\mu = 0$

for all leafwise smooth continuous function u. Observe that uniqueness implies in particular that the foliation $\mathcal{W}^s$ is ergodic with respect to μ [**G**].

Denote C_1 the space of Hölder continuous stable 1-forms α which are closed, are of class C^1 along the leaves of the foliation $\mathcal{W}^s$ and such that $\delta_s\alpha$ is Hölder continuous on SM. Two forms α, α' in C_1 are said to be cohomologous if there is a Hölder continuous function u on SM such that u is C^1 along the leaves of $\mathcal{W}^s$ and $\alpha - \alpha' = d_s u$. Denote H^1 the quotient space of cohomology classes of closed 1-forms in C_1.

Theorem 1. *In each cohomology class $[\alpha]$ in H^1 there is a unique harmonic form $\bar{\alpha}$. The assignement $\alpha \to \bar{\alpha}$ is linear and $\sigma([\alpha]) = \int \|\bar{\alpha}\|^2 d\mu$ defines a positive definite quadratic form on H^1.*

1.2 Examples of stable 1-forms

If the form α in C_1 is lifted from a closed 1-form on M, then by the Hodge-de Rham theorem and the uniqueness in Theorem 1, the harmonic 1-form $\bar{\alpha}$ is lifted from a harmonic 1-form on M and

$$\sigma([\alpha]) \leq \frac{1}{\text{vol } M} \text{ energy } \alpha,$$

with equality only when α is harmonic.

Other elements in C_1 are defined by the global geometry of leaves. Define e.g. α_0 by $\alpha_0 = d_s b$. Let X_v be the geodesic spray on SM, i.e. the vector field generating the geodesic flow. Then $\alpha_0^\sharp = -X_v$ and $\delta_s\alpha_0 = div_s X_v$ is the mean curvature of the stable horosphere.

Let $G(v,w)$ be the Green function associated to the Laplace-Beltrami operator on the leaves. Define $k(v,w)$ if the leaf is simply connected, by (see [**AS**], [**A1**]):

$$k(v,w) = \lim_{s\to\infty} \frac{G(w, \phi_{s+b(v,w)}w)}{G(v, \phi_s v)}$$

set $\alpha_1 = d_s \log k$ if the leaf is simply connected and extend α_1 on the whole SM by continuity. By [**H, lemma 3.2.**], α_1 is Hölder continuous on SM and since $\delta_s\alpha_1$ is the function $\|\alpha_1\|^2$, $\alpha_1 \in C_1$. Recall that for any closed stable 1-form

$$\int \delta_s \alpha d\mu = \int < \alpha, \alpha_1 >_{gs} d\mu$$

and that

$$\int \|\alpha_1\|^2 d\mu \leq \sigma([\alpha_1])$$

with equality if and only if (M, g) is asymptotically harmonic [**L3**].

Let now φ be any continuous function on SM, which is C^2 along the leaves and with Hölder continuous 2-jets. Set for w in $W^s(v)$

$$A_\varphi(v,w) = \lim_{s\to\infty} \int_0^{s+b(v,w)} \varphi(\phi_u v)du - \int_0^s \varphi(\phi_u v)du \ .$$

Then $\alpha_\varphi = d_s A_\varphi$ is the unique form in C_1 such that $\alpha_\varphi(X_v) = \varphi(v)$. The above construction applied to the function

$$J^s(v) = -\frac{d}{dt} \log |\det D\phi_{t/TW^s(v)}|\Big|_{t=0}$$

yields another element of C_1, denoted α_2, characterized by $\alpha_2(X_v) = J^s(v)$.

The forms α_0, α_1 and α_2 respectively are closely associated with the Bowen Margulis measure of maximal entropy for the geodesic flow on SM, the invariant harmonic measure and the Liouville measure respectively. Any two of these measures coincide if and only if the corresponding 1-forms are cohomologous.

1.3 Periods and thermodynamic formalism

Theorem 2. *Two forms in C_1, α and α', are cohomologous if and only if $\int_\gamma(\alpha - \alpha') = 0$ for all closed curves γ in W^s.*

By theorem 2, a cohomology class in H^1 is completely described by its periods, i.e. the values of $\int_\gamma \alpha$ when α runs through all periodic orbits of the geodesic flow. Since these integrals are in fact given by

$$\int_\gamma \alpha = \int_0^{\ell(\gamma)} \alpha(X_{\phi_u v})du$$

for some v in γ, thermodynamic formalism of the Hölder continuous function $\alpha(X_v)$ applies verbatim to elements in H^1. For instance define the Pressure function on H^1 by

$$P([\alpha]) = P(\alpha) = \lim_{T\to\infty} \frac{1}{T} \log \sum_{\{\gamma = \ell(\gamma) \leq T\}} e^{\int_\gamma \alpha}.$$

The pressure P is a convex function on H^1, real analytic on finite dimensional subspaces ([**R**]) and satisfies:

$P(\alpha + a\alpha_0) = P(\alpha) + a,$

$P(0)$ is the topological entropy of the geodesic flow, and

$P(-\alpha_1) = P(-\alpha_2) = 0$ (see respectively [L1] and [BR]).

Theorem 3. *There is a positive number ℓ such that for all α in C_1*

$$\int \delta_s \alpha d\mu = \ell \frac{d}{ds} P(-\alpha_1 + s\alpha)|_{s=0} \tag{1}$$

and if $\int \delta_s \alpha d\mu = 0$

$$\sigma([\alpha]) = \frac{\ell}{2} \frac{d^2}{ds^2} P(-\alpha_1 + s\alpha)|_{s=0} \tag{2}$$

From the above relation follows that $\ell = \int \delta_s \alpha_0 d\mu$ is the average speed of escape to infinity of the Brownian notion on the universal cover $(\widetilde{M}, \tilde{g})$ of (M, g) (see [**K**]). Theorem 3 expresses that σ defines a positive definite quadratic form on the tangent plane at $-\alpha_1$ to the set $\{P = 0\}$.

As will be explained below, Theorem 1 follows from [**L3**] and Theorem 2 is a reformulation of Livsic Theorem [**Li**]. The main idea behind the proof of Theorem 3 is an observation by Y. Le Jan [**LJ**] that the limit behavior of a closed 1-form evaluated on the Brownian path is the same as the limit behavior of the same closed 1-form evaluated on a nearby geodesic path. The average behavior is given by the invariant harmonic measure [**L1**], and since this measure is the equilibrium measure for $-\alpha_1$ formula (1) follows from general thermodynamic formalism. The variance for an equilibrium measure has been computed by Ruelle ([**R**], page

99, see also [**K-S**]) and comparaison will yield formula (2) in theorem 3.

II. The stable foliation

2.1 Description

Let $\widetilde{M}$ be the universal cover of $M, \partial\widetilde{M}$ the geometric boundary of $\widetilde{M}$ and Γ the fundamental group of M. The group Γ acts on $\widetilde{M}$ and on $\partial\widetilde{M}$. The quotient space $\widetilde{M} \times \partial\widetilde{M}/\Gamma$ will be identified with the unit tangent bundle of M or with $M_0 \times \partial\widetilde{M}$ where M_0 is a fundamental domain for the action of Γ on $\widetilde{M}$. The identification of SM with $M_0 \times \partial\widetilde{M}$ consists in lifting a unit vector v in SM in a unit vector $\tilde{v}$ in SM_0 and in associating to $\tilde{v}$ the pair (x, ξ) where x is the footpoint of $\tilde{v}$ and ξ is the point at infinity of the geodesic ray

$$\{\gamma_{\tilde{v}}(t), t \geq 0\}$$

defined by

$$(\gamma_{\tilde{v}}(0), \dot{\gamma}_{\tilde{v}}(0)) = \tilde{v}.$$

There is a natural Riemannian metric on SM the Sasaki metric. There are natural metrics on $M_0 \times \partial\tilde{M}$, defined by choosing a point x_0 in M_0 and a conformal distance at infinity d,

$$d(\xi, \eta) = exp - \epsilon(\xi/\eta)_{x_0},$$

where $(\xi/\eta)_{x_0}$ is the Gromov product, see e.g. [**GH**], and $\epsilon > 0$ is sufficiently small. The above identification is Hölder continuous in both directions with respect to these metrics, so that the property of being Hölder continuous for some exponent is the same in all these metrics.

Define the stable leaf $\widetilde{W}^s(\tilde{v})$ of a point $\tilde{v}$ in $S\widetilde{M}$ by:

$$\widetilde{W}^s(\tilde{v}) = \left\{ \begin{array}{lll} \tilde{w}: & \text{there is a real} b(\tilde{v}, \tilde{w}) & \text{such that} \\ d(\phi_s\tilde{v}, \phi_{s+b(\tilde{v},\tilde{w})}\tilde{w}) \longrightarrow 0 \ . & & \end{array} \right\}.$$

Observe that through the above identification

$$\widetilde{W}^s(x, \xi) = \widetilde{M} \times \{\xi\}$$

The set $\widetilde{W}^s(\tilde{v})$ is a lift of $W^s(v)$ and the metric g_s, the operators d_s , δ_s and Δ_s lift to the natural metric $\tilde{g}$ on $\widetilde{M} \times \{\xi\}$ and to the operators

d, δ and Δ defined with $(\widetilde{M} \times \{\xi\}, \tilde{g})$. In particular the leaf $W^s(v)$ is isometric to $\widetilde{W}(\tilde{v})/\operatorname{Stab}\xi$. We have either $\operatorname{Stab}\xi = \{e\}$ and then $W^s(v)$ is isometric to $\tilde{M}$ or $\operatorname{Stab}\xi \simeq \mathbb{Z}$. This happens if and only if $W^s(v) = W^s(w)$ where w belongs to a periodic orbit γ and then $W^s(v)$ is isometric to $\widetilde{M}/\mathbb{Z}_\gamma$, where $\mathbb{Z}_\gamma$ is one of the cyclic groups which represent γ.

2.2 Proof of Theorem 1

Consider α a 1-form in C_1 and the function $\psi = \delta_s\alpha - \int \delta_s\alpha d\mu$. The function ψ is Hölder continuous on SM and has integral zero with respect to the measure μ. By [**L3**] corollary 1, there exists a Hölder continuous function U on SM, C^2 along the leaves of $\mathcal{W}^s$, unique up an additive constant and such that $\Delta_s U = \psi$. Set $\bar{\alpha} = \alpha + d_s U$. The form $\bar{\alpha}$ is closed, cohomologous to α and satisfies

$$\Delta_s\bar{\alpha} = -d_s\delta_s\bar{\alpha} = -d_s(\delta_s\alpha - \Delta_s U) = 0$$

since $\delta_s\alpha - \Delta_s U$ is a constant function.

To prove that $\bar{\alpha}$ is unique with these properties, take $\bar{\alpha}'$ harmonic and cohomogous to α. Then there is a function U' satisfying $\bar{\alpha} - \bar{\alpha}' = d_s U'$ and $d_s\delta_s d_s U' = 0$. Observe that $\bar{\alpha}$ and $\bar{\alpha}'$ are smooth along the leaves of $\mathcal{W}^s$ and that therefore the function $\Delta_s U' = \delta_s\bar{\alpha}' - \delta_s\bar{\alpha}$ is smooth along the leaves. Since the function $\Delta_s U'$ satisfies $d_s(\Delta_s U') = 0$, the function $\Delta_s U'$ is constant along the leaves, i.e. constant $\mu a.e.$ by ergodicity of the foliation. The value of this constant is 0 since $\int \Delta_s U' d\mu = 0$. Since the function U' satisfies $\Delta_s U' = 0\mu$ *a.e.*, the function U' is constant on μ, *a.e.* leaf. Finally since the function U' is continuous and the foliation $\mathcal{W}^s$ is transitive, the function U' is constant and $\bar{\alpha}' = \bar{\alpha}$. In particular $\bar{\alpha} = 0$ if and only if α is exact.

By construction, the application $\alpha \to \bar{\alpha}$ is linear from C_1 to H^1. Moreover if $\sigma[\alpha]$ vanishes, $\bar{\alpha} = 0$ and α is exact the quadratic form σ is positive definite on H^1.

2.3 Proof of theorem 2

By definition if two closed 1-forms α and α' are cohomologous, then

$\int_\gamma(\alpha - \alpha') = 0$ for any closed curve γ living in a single leaf, in particular, for any periodic orbit.

Conversely assume that α is a stable form in C_1 such that $\int_\gamma \alpha = 0$ for any periodic orbit γ. Then the Hölder continuous function $\psi(v) = \alpha(X_v)$ is such that $\int_0^{\ell(\gamma)} \psi(\phi_s v)ds = 0$ for all v belonging to a periodic orbit γ of length $\ell(\gamma)$. By Livsic Theorem [**Li**] there exists a Hölder continuous U on SM such that U is smooth along the trajectories of the geodesic flow and satisfies:

$$\mathcal{L}_X U(v) = \alpha(X_v).$$

We claim that the function U is C^1 along the leaves and that $d_s U = \alpha$ (we follow [**LMM lemma 2.2**]): in order to differentiate U in the direction of a vector field Y tangent to W^s, choose first v in the stable manifold of some periodic orbit of period τ. Then there are numbers

$$T_n, T_{n+1} - T_n \to \tau$$

and some v_0 in the periodic orbit such that $(\phi_{T_n} v)_n$ converge towards v_0 as $n \to \infty$, Then,

$$U(v) = U(v_0) - \lim_{n\to\infty} \int_0^{T_n} \alpha(X_{\phi_s v})ds.$$

Let ψ be the flow associated with Y locally on $W^s(v)$ and fix s_0 small. Then $\psi_{s_0} v$ is another point of $W^s(v)$ and there are numbers $T'_n, T'_{n+1} - T'_n \to \tau$ such that $(\phi_{T'_n} \psi_{s_0} v)_n$ converge towards the same point v_0 and

$$U(\psi_{s_0} v) = U(v_0) - \lim_{n\to\infty} \int_0^{T'_n} \alpha(X_{\phi_s \psi_{s_0} v})ds$$

Finally, since α is a closed form with

$$\int_0^{\tau} \alpha(\phi_s v_0)ds = 0$$

$$\begin{aligned} U(\psi_{s_0} v) - U(v) &= \lim_{n\to\infty} \int_0^{T_n} \alpha(X_{\phi_s v})ds - \int_0^{T'_n} \alpha(X_{\phi_s \psi_{s_0} v})ds \\ &= \int_0^{s_0} \alpha(Y_{\psi_t v})dt \end{aligned}$$

The formula extends by continuity to the whole SM, and shows that $d_s U = \alpha$.

2.4 Higher degrees

Stable leaves have no other non trivial topology. The following proposition is therefore natural.

Proposition . *Let α be a closed stable p-form of class C^1 with Hölder continuous 1-jets and $p > 1$. Then α is exact, there exists a stable $(p-1)$ form of class $C^1 \beta$ such that $d_s\beta = \alpha$.*

In fact, since the flow is Anosov, there are constants $C > 0$ and $\lambda < 1$ such that for all $t > 0$ all vector Y in $T_v W^s \cap X_v^\perp$:

$$\|\phi_{t_*} Y\|_{\phi_t v} \leq C\lambda^t \|Y\|_v$$

Therefore

$$\|\phi_t^* \alpha\| \leq C^{p-1} \lambda^{(p-1)t} \|\alpha\|$$

for any stable form. Since $p > 1$, the following integral makes sense and defines a stable $(p-1)$ form β of class C^1 with Hölder continuous 1 jets:

$$\beta = -\int_0^\infty i_X(\phi_t^* \alpha) dt$$

Then

$$\begin{aligned} d_s\beta &= -\int_0^\infty d_s(i_X \phi_t^* \alpha) dt \\ &= \int_0^\infty (i_X d_s \phi_t^* \alpha - \mathcal{L}_X \phi_t^* \alpha) dt \\ &= -\int_0^\infty \mathcal{L}_X \phi_t^* \alpha \; dt \quad \text{since} \quad d_s \phi_t^* \alpha = 0 \\ &= \alpha \; . \end{aligned}$$

III. Ergodic theory of the stable foliation and proof of Theorem 3

3.1 Stable Brownian motion (this section is taken from [**G**], [**L2**], [**L3**]). Consider the leafwise Laplace operator Δ_s. There is a leafwise Brownian motion associated to it, i.e. a family of probability measures $\mathbb{P}_v$ on the space $C(\mathbb{R}_+, SM)$ of continuous paths in SM such that with probability 1, $\omega(0) = v$ and $\omega(t) \in W^s(v)$ for all $t \geq 0$ and such that for all $v, (C(\mathbb{R}_+, SM), \mathbb{P}_v)$ is a Markov process with generator Δ_s. The proof of the uniqueness of the Δ_s harmonic measure also gives a description of μ :

$$\int_{SM} f d\mu = \int_{M_0} \left(\int_{\partial \widetilde{M}} f(y, \xi) d\tilde{\mu}_y(\xi) \right) dm_0(y) ,$$

where m_0 is the normalized Lebesgue measure on the fundamental domain M_0 and $\tilde{\mu}_y$ is the harmonic measure of the point y on the boundary at infinity $\partial\widetilde{M}$.

The other description of the measure μ uses several properties of the geometry of the stable foliation. There is a local product structure given by the stable foliation $\mathcal{W}^s$ and the strong unstable foliation $\mathcal{W}^{uu}$, that is the manifold SM can be covered by the ranges of charts

$$\varphi \colon B^{n+1} \times B^n \to SM,$$

where B^q is the unit ball in $\mathbb{R}^q$, φ is Hölder continuous and such that, for all x, y, $\varphi(B^{n+1} \times \{y\})$ is a neighborhood of $\varphi(0, y)$ in $W^s(\varphi(0, y))$, $\varphi(\{x\} \times B^n)$ is a neighborhood of $\varphi(x, 0)$ in $W^{uu}(\varphi(x, 0))$. There is a family of measures μ^{uu} on the unstable manifolds such that

$$d\tau\mu^{uu} = k(v, \tau v) d\mu^{uu}$$

whenever τ is an invertible measurable mapping from a subset of $W^{uu}(v)$ to a subset of $W^{uu}(\tau v)$ with $\tau w \in W^s(w)$ for all w ([**L1**]). Then the measure

$$f \to \int_{\varphi(\{0\} \times B^n)} d\mu^{uu} \left(\int_{\varphi(B^{n+1} \times \{y\})} f(\varphi(x, y))\, k(\varphi(0, y), \varphi(x, y)) d\,\mathrm{vol}_{g^s} \right)$$

do not depend on the chart φ on its support and globally defines a finite measure $\bar{\mu}$. By construction, the measure $\bar{\mu}$ is Δ_s harmonic (see [**G**]) and therefore $\mu = \frac{\bar{\mu}}{\bar{\mu}(SM)}$.

Write $-$ for the antipodal map on SM. Then $-W^s(-v)$ is the unstable manifold $W^u(v)$ and the mapping is an isometry between

$$(W^s(-v), g_s) \text{ and } (W^u(v), g_u),$$

where g_u is defined analogously. The measure $-\mu$ is the unique Δ_u harmonic probability measure. We write μ^{ss} for the corresponding family of measures on strong stable manifolds W^{ss} given by $\mu^{ss} = -\mu^{uu}$.

3.2 Invariant harmonic measure

There is also a unique probability measure m on SM which is invariant under the geodesic flow and such that, in local charts, the conditional measures of m with respect to W^{uu} are equivalent to μ^{uu} [**L1**]. From the description of m in [**H**] follows that on the range of a local chart, m is equivalent to the product of the measures μ^{ss} and some measure μ^{u} on local unstable manifolds, with a positive Hölder continuous positive density. Recall also that m is the Gibbs measure of the function $-X \log k(v,\cdot) = -\alpha_1(X_v)$. Therefore for any continuous function f on SM

$$\int f dm = \frac{d}{ds} P(-\alpha_1(X) + sf)_{|s=0}$$

In particular for $f = \alpha(X)$ for some α in C_1;

$$(*) \qquad \int \alpha(X) dm = \frac{d}{ds} P(-\alpha_1 + s\alpha)_{|s=0}$$

The set of stable forms on C_1 such that $\int \alpha(X)dm = 0$ can therefore be seen as a tangent plane to the convex hypersurface $P = 0$.

Given a Hölder continuous function f on SM with $\int f dm = 0$, we have ([**KS**])

$$\lim_{t\to\infty} \frac{1}{t} \int \left(\int_0^t f \circ \phi_r dr \right)^2 dm = \frac{d^2}{ds^2} P(-\alpha_1(X) + sf)_{|s=0}$$

Therefore, for any α in C_1 with $\int \alpha(X)dm = 0$,

$$(**) \qquad \lim_{t\to\infty} \frac{1}{t} \int \left(\int_0^t \alpha(X_{\phi_r} v) dr \right)^2 dm = \frac{d^2}{ds^2} P(-\alpha_1 + s\alpha)_{|s=0} \cdot$$

3.3 Proof of theorem 3 (1)

Consider the stable Brownian Motion $(C(\mathbb{R}_+, SM), \mathbb{P}_v)$ of section 3.1. Each ω in $C(\mathbb{R}_+, SM)$ can be identified with $(\tilde{\omega}, \xi_0)$ where $\tilde{\omega}$ is a trajectory of the Brownian Motion on $\widetilde{M}$ starting at x_0 and ξ_0 is such that $\omega_0 = (x_0, \xi_0)$.

For $\mathbb{P}_v$ *a.e.* ω, the trajectory $\tilde{\omega}$ converges towards some point at infinity $\eta(\omega)$ and let γ_ω denote the geodesic in $\widetilde{M}$ such that

$$\gamma_\omega(+\infty) = \xi_0, \gamma_\omega(-\infty) = \eta(\omega)$$

and

$$b((x_0, \xi_0), (\gamma_\omega(0), \xi_0)) = 0 \ .$$

Set

$$\tilde{v}(\omega) = (\gamma_\omega(0) \ , \dot{\gamma}_\omega(0))$$

and denote for all ω such that γ_ω is defined and all $t \geq 0, \zeta_t(\omega)$ the real number such that $\gamma_\omega(\zeta_t(\omega))$ is the closest point to $\tilde{\omega}(t)$ on the geodesic γ_ω. Let $v(\omega)$ be the projection of $\tilde{v}(\omega)$ on SM. This construction associates to each trajectory ω in $C(\mathbb{R}_+, SM)$ a vector $v(\omega)$ in $W^{ss}(\omega_0)$ and a real process $\zeta_t(\omega)$ with the following properties: for $\mathbb{P}_v$ *a.e.* ω,

$$\limsup_t \frac{1}{\log t} d(\tilde{\omega}_t, \gamma_\omega(\zeta_t(\omega)) \leq C$$

(see [**A2**], théorème 7.3) and the distribution m_{ω_0} of $v(\omega)$ has a Hölder continuous density with respect to μ^{ss}.

Consider now a stable form α in C_1 and the real process $(Y_t)_{t \geq 0}$,

$$Y_t = \int_{\omega(0,t)} \alpha.$$

By [**L3, Corollary 2**], there is a Hölder continuous function U on SM such that

$$(Y_t + t \int \delta_s \alpha d\mu + U(\omega_t) - U(\omega_o))_{t \geq 0}$$

is a $\mathbb{P}_v$ martingale with increasing process $2\bar{\alpha}(\omega_s)ds$. The same result applied to α_0 yields a Hölder continuous function U_0 such that

$$(b((\tilde{\omega}_0, \xi_0), (\tilde{\omega}_t, \xi_0)) + t \int \delta_s \alpha_0 d\mu + U_0(\omega_t) - U(\omega_0))_{t \geq 0}.$$

is a $\mathbb{P}_v$ martingale with increasing process $2\bar{\alpha}_0(\omega_s)ds$.

Therefore we have on a set of $\mathbb{P}_v$ measure 1:

i) $\lim\limits_{t \to \infty} \frac{1}{\log t} |Y_t(\omega) + \int_{\zeta_t(\omega)}^0 \alpha(\dot{\gamma}_\omega(u))du| \leq C$,

ii) $\lim_{t\to\infty} \frac{1}{\log t}|b((\tilde{\omega}_0, \xi_0), (\tilde{\omega}_t, \xi_0)) - \varsigma_t(\omega)| \leq C$,

iii) $\lim_{t\to\infty} |\frac{Y_t}{t}(\omega) + \int \delta_s \alpha d\mu| = 0$ and

iv) $\lim_{t\to\infty} |\frac{b((\tilde{\omega}_0,\xi_0)(\tilde{\omega}_t,\xi_0))}{t} + \int \delta_s \alpha_0 d\mu| = 0$.

Furthermore the distribution under $\mathbb{P}_v$ of

v) the variable $\frac{Y_t + t\int \delta_s \alpha d\mu}{\sqrt{t}}$ is asymptotically normal with variance $2\sigma([\alpha])$.

vi) the variable

$$\frac{1}{\sqrt{t}}(b(\tilde{\omega}_0, \xi_0)(\tilde{\omega}_t, \xi_0) + t\int \delta_s \alpha_0 d\mu)$$

is asymptotically normal with variance $2\sigma([\alpha_0])$.

Properties i) and ii) are obtained by writing the integral of the closed forms α and α_0 on the Brownian path $\omega(0,t)$ as the integral on a continuous, piecewise smooth curve $c = (c_1, c_2, c_3)$ where c_1 lies is $W^{ss}(x_0, \xi_0)$, c_2 is the geodesic $\gamma_\omega(0, \varsigma_t(\omega))$ and c_3 is the curve $\{(\gamma_3(s), \xi_0)\}$ where $\gamma_3(s)$ is the geodesic between $\gamma_\omega(\varsigma_t(\omega))$ and $\tilde{\omega}_t$.

If $(Z_t)_{t\geq 0}$ is a real process on a probability space $(\Omega\ ,\ \mathbb{P})$, we say that the distribution of Z_t is asymptotically normal under $\mathbb{P}$ with variance $\sigma^2 > 0$ if for all real λ

$$\lim_{t\to\infty} \int e^{i\lambda \mathbb{Z}_t} d\mathbb{P} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{\lambda^2}{2\sigma^2}}$$

Property v) and vi) follow from the central limit theorem for martingales and from the fact that as $t \to \infty$

$$\int \left(\frac{1}{t}\int_0^t ||\bar{\alpha}(\omega_s)||^2 ds - \int ||\bar{\alpha}||^2 d\mu\right) d\mathbb{P}_{x_0\xi_0} \to 0$$

$$\int \left(\frac{1}{t}\int_0^t ||\bar{\alpha}_0(\omega_s)||^2 ds - \int ||\bar{\alpha}_0||^2 d\mu\right) d\mathbb{P}_{x_0\xi_0} \to 0$$

(cf the proof of Theorem 1 in ([**L3**]))

We now translate properties i) to vi) into properties of $v(\omega)$ and $\varsigma_t(\omega)$ under $\mathbb{P}_{\omega_0}$, where the vector $v(\omega)$ and the process $\zeta_t(\omega)$ are given by the above construction:

a) the distribution m_{ω_0} of $v(\omega)$ has a positive Hölder continuous density with respect to μ^{ss},

b) $\lim_{t\to\infty} \frac{\varsigma_t(\omega)}{t} = -\ell$ for $\mathbb{P}_v$ $a.e\ \omega$,

c) the distribution of $\frac{1}{\sqrt{t}}(\varsigma_t(\omega) + t\ell)$ is asymptotically normal with variance $2\sigma([\alpha_0])$,

d) $\lim_{t\to\infty} \frac{1}{t} \int_{\varsigma_t(\omega)}^0 \alpha(\dot{\gamma}_\omega(u))du = \int \delta_s \alpha d\mu$ where $(\gamma_\omega(0), \dot{\gamma}_\omega(0)) = \tilde{v}(\omega)$, and

e) the distribution of

$$\frac{1}{\sqrt{t}}(\int_{\varsigma_t(\omega)}^0 \alpha(\dot{\gamma}_\omega(u))du - t\int \delta_s \alpha d\mu)$$

is asymptotically normal with variance $2\sigma([\alpha])$.

In c) and e) we used the fact that if (X_t) is asymptotically normal and (Z_t) converges in probability to zero, then $(X_t + Z_t)$ is asymptotically normal with the same variance.

The first relation in theorem 3 now follows from a), b) d), (*) and the following lemma 1, applied to the function ψ given by $\psi(v) = \alpha(X_v)$:

Lemma 1. *For any continuous function ψ, m_ω a.e. v.*

$$\lim_{T\to\infty} \frac{1}{T} \int_{-T}^0 \psi(\phi_u v)du = \int \psi dm \ .$$

Proof of lemma 1. write B for the set of v in SM such that

$$\lim_{T\to\infty} \frac{1}{T} \int_{-T}^0 \psi(\phi_u v)du = \int \psi dm \ .$$

By the ergodic theorem, the set B has measure 1. Since the function ψ is continuous, the set B is a union of unstable leaves. By the product structure of the measure m, for any strong stable leaf, we have

$$\mu^{ss}(B^c \cap W^{ss}) = 0.$$

Since the measure m_{ω_0} is absolutely continuous with respect to μ^{ss}, the lemma follows.

3.4 Proof of theorem 3 (2)

Assume now that $\int \delta_s \alpha d\mu = 0$ and set $\psi(v) = \alpha(X_v)$. From e) follows that under $\mathbb{P}_v$ the distribution of the variable:

$$\frac{1}{\sqrt{t}} \int_{-\ell t}^0 \psi(\phi_s v(\omega))ds + \frac{1}{\sqrt{t}} \int_{\varsigma_t(\omega)}^{-\ell t} \psi(\phi_s v(\omega))ds$$

is asymptotically normal with variance $2\sigma([\alpha])$. Like in 3.3, theorem 3 (2) will follow from comparaison of this property with $(**)$. We first have

Lemma 2. *Let ψ be a continuous function on SM, $\int \psi dm = 0$. Then for all $\epsilon > 0$*

$$\lim_{t\to\infty} \mathbb{P}_v(\omega : |\int_{\varsigma_t(\omega)}^{-\ell t} \psi(\phi_u v(\omega))du| \geq \epsilon\sqrt{t}) = 0 \ .$$

The proof of Lemma 2 is parallel to the proof of Lemma 1: with large probability $\frac{1}{\sqrt{t}}\int_{\varsigma_t(\omega)}^{-\ell t}\psi$ is small since $\int \psi dm$ is zero and $\varsigma_t(\omega) + \ell t$ is comparable to $\sqrt{t}$. More explicitly, there is $k > 0, T_0$ such that for all $t \geq T_0$

$$\mathbb{P}_v\{\omega\colon \frac{\sqrt{t}}{k} \leq |\varsigma_t(\omega) + \ell t| \leq k\sqrt{t}\} \geq 1 - \frac{\epsilon}{2} \ .$$

Also fix a finite cover of SM by open sets $\mathcal{O}_i$ which are ranges of (W^u, W^{ss}) charts. There is a number $\delta > 0$ such that if B is a union of subsets $B_i, B_i \subset \mathcal{O}_i, B_i$ is a union of W^u plaques with $m(B) \geq 1 - \delta$ then $m_{x_0,\xi_0}(B) \geq 1 - \frac{\epsilon}{2}$ for all (x_0, ξ_0).

Choose T_1 by the ergodic theorem such that $m(A_{T_1}) \geq 1 - \delta$, where

$$A_{T_1} = \{v\colon \forall t, t \geq T_1 \ , |\frac{1}{t}\int_0^t \psi(\phi_u v)du| \leq \frac{\epsilon}{2k}$$
$$\text{and } |\frac{1}{t}\int_{-t}^0 \psi(\phi_u v)du| \leq \frac{\epsilon}{2k}\}.$$

By continuity of ψ there is a T_2 such that if $t \geq T_2$ if v_1 and v_2 are in the same W^u plaque in the same $\mathcal{O}_i$, then for all s, $\frac{1}{k}\sqrt{t} \leq s \leq k\sqrt{t}$

$$\frac{1}{s}|\int_{-\ell t}^{-\ell t+s} \psi(\phi_u v_1)du - \int_{-\ell t}^{-\ell t+s} \psi(\phi_u v_2)du| \leq \frac{\epsilon}{2k}$$

and for all s , $s \leq -\frac{1}{k}\sqrt{t}$

$$\frac{1}{|s|}|\int_{-\ell t+s}^{-\ell t} \psi(\phi_u v_1)du - \int_{-\ell t+s}^{-\ell t} \psi(\phi_u v_2)du| \leq \frac{\epsilon}{2k}.$$

For t bigger than $T_0, k^2T_1^2, T_2$, write B_t for the set of v in SM such that there is v_1 in the same W^u plaque as v with $\Phi_{-\ell t}v_1 \in A_{T_1}$. Then for all

v in B_t, all s with $\frac{1}{k}\sqrt{t} \le |s| \le k\sqrt{t}$,

$$|\frac{1}{s}\int_{s-\ell t}^{-\ell t} \psi(\phi_u v)du| \le \frac{\epsilon}{k} \ .$$

By definition of B_t,

$$m(B_t) \ge m(\phi_{\ell t} A_{T_1}) \ge 1-\delta,$$

so that

$$\mathbb{P}_v\{\omega, v(\omega) \in B_t\} \ge 1 - \frac{\epsilon}{2}.$$

Lemma 2 follows since when $v(\omega) \in B_t$ and

$$\frac{\sqrt{t}}{k} \le |\xi_t(\omega) + \ell t| \le k\sqrt{t},$$

then

$$|\int_{\xi_t(\omega)}^{-\ell t} \psi(\phi_u v(\omega)du)| \le |\varsigma_t(\omega) + \ell t|\frac{\epsilon}{k} \le \epsilon\sqrt{t} \ .$$

From lemma 2 we conclude that for all (x_0, ξ_0) under $m_{(x_0,\xi_0)}$, the variable

$$\frac{1}{\sqrt{t}}\int_{-\ell t}^{0} \psi(\phi_u v)du$$

is asymptotically normal with the same variance $2\sigma([\alpha])$. It follows that under the probability m',

$$m' = \int m_{(x_0,\xi_0)} dm(x_0, \xi_0),$$

the variable

$$\frac{1}{\sqrt{t}}\int_{-\ell t}^{0} \psi(\phi_u v)du$$

is asymptotically normal with variance $2\sigma([\alpha])$. Observe now that the measure m' has a continuous density with respect to m as can be checked readily in local charts.

Lemma 3. *Let $(\mathcal{X}, \mathcal{A}, m; \phi_t)$ be an ergodic flow, ψ a bounded function, U a non negative function such that $\int U dm = 1$. Assume that under $m' = Um$ the variable $\frac{1}{\sqrt{t}}\int_0^t \psi\phi_u du$ is asymptotically normal with variance σ. Then the same is true under m.*

To prove Lemma 3, observe that the set of functions U such that, under the measure Um, the variable $\frac{1}{\sqrt{t}}\int_0^t \psi\phi_u du$ is asymptotically normal with variance σ is clearly invariant under ϕ, convex, and closed in L^1 norm. By the ergodic theorem, if it contains one function U, it also contains the constant 1.

Using lemma 3 and the discussion above we get

Corollary . *Let ψ be a continuous function on SM, of class C^2 along the leaves, with Hölder continuous 2-jets. Then under the invariant harmonic measure m. The variable*

$$\frac{1}{\sqrt{t}}(\int_0^t \psi(\phi_u v)du - t\int \psi dm)$$

is asymptotically normal. If we write

$$\psi(v) = \int \psi dm + \alpha(X_v)$$

for α in C_1, the limit variance is $\frac{2}{\ell}\sigma([\alpha])$.

The relation (2) in theorem 3 follows by comparing the value of the limit variance in the corollary and in (**).

Observe also that, in the same way as in [**LJ**], we prove a central limit theorem for the averages of regular functions along the geodesic flow and the harmonic invariant measure. This is a particular case of a result of Ratner ([**Ra**]).

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François Ledrappier
c.n.r.s. u.r.a. 169
Centre de Mathématiques,
Ecole Polytechnique
91128 Palaiseau
France